On some problems concerning the distribution of polynomials with bounded roots

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Around 1917 Issai Schur introduced what is now called the d-dimensional *Schur-Cohn region* $\mathcal{E}_d$, defined as the set of all coefficient vectors of monic real polynomials of degree d each of whose roots lies in the open unit disk. In 1922 Cohn developed an algorithm to check if a vector belongs to $\mathcal{E}_d$, and in the following decades further properties of the Schur-Cohn region were published.

In the 1970s Fam calculated the d-dimensional Lebesgue measure $\lambda_d(\mathcal{E}_d)$ and proved that

$$v_d := \lambda_d(\mathcal{E}_d) = 2^d \prod_{j=1}^{\lfloor d/2 \rfloor} \left(1 + \frac{1}{2j}\right)^{2j-d}.$$

If $P \in \mathbb{R}[X]$ is a polynomial of degree d with r real and $2s$ nonreal roots (r, s) is called the *signature* of P. Akiyama and Pethő studied the subsets $\mathcal{E}_d^{(s)}$ of polynomials with fixed signature $(d-2s, s)$ in $\mathcal{E}_d$ and their volumes $v_d^{(s)} := \lambda_d(\mathcal{E}_d^{(s)})$. Using relations of $v_d^{(s)}$ with Selberg integrals and their generalizations by Aomoto they derived

$$v_d^{(0)} = 2^{d(d+1)/2} \prod_{j=1}^{d} \frac{(j-1)!^2}{(2j-1)!}.$$

Furthermore they proved that the numbers $v_d^{(s)}$ are rational and that $v_d/v_d^{(0)}$ is an odd integer for all $d \geq 1$. Following numerical experiments, they stated the

Conjecture. The quotient $v_d^{(s)}/v_d^{(0)}$ is an integer for all $s \leq d/2$.

In particular, for the special instance $d = 2s$ they conjectured $v_{2s}^{(s)}/v_{2s}^{(0)} = 2^{2s(s-1)}\binom{2s}{s}$.

In the talk we will present the original and a more recent proof for explicit formulæ for $v_d^{(s)}/v_d^{(0)}$ for arbitrary $s \leq d/2$ and, as a by-product, verify the conjectures from above. The ingredients of our proofs comprise Selberg type integrals, the Cauchy double alternant, Pfaffians and principal minors of partial Hilbert matrices.